Lipschitz determinacy and Arithmetic Transfinite Recursion

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Abstract. We investigate the logical strength of Lipschitz determinacy, and the tightly related Semi-Linear Ordering principle, for the first levels of the Borel hierarchy in the Baire space. As a result, we obtain characterizations of ATR_0 in terms of these determinacy principles.
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1 Introduction

Reverse mathematics is a well-established research program in Mathematical Logic motivated by the fundamental question: *Which set existence axioms are needed to prove the known theorems of mathematics?* As a contribution to this program, we study in the context of second order arithmetic the logical strength of the Lipschitz determinacy principle Det_L and the Semi-Linear Ordering principle SLO_L restricted to the first levels of the Borel hierarchy in the Baire space. In our main result we characterize ATR_0 (one of the "Big Five" theories widely studied in second order arithmetic) by using these determinacy principles.

Lipschitz games were first introduced in the setting of descriptive set theory by W.W. Wadge [12] as a tool for studying the relative complexity of subsets of the Baire space ω^ω. Given $A, B \subseteq \omega^\omega$, A is said to be Lipschitz reducible to B, in symbols $A \leq_L B$, if there is a Lipschitz function F such that $x \in A$ if, and only if, $F(x) \in B$ (note that $\leq_L$ is a natural analog of the many-one reducibility of computability theory). Wadge proved that $\leq_L$ can be studied in terms of *Lipschitz games*. The Lipschitz game $G_L(A, B)$ is the game on ω where players I and II alternatively play natural numbers a_i and b_i, and player II wins just in case $\langle a_0, a_1, a_2, \dots \rangle \in A \Leftrightarrow \langle b_0, b_1, b_2, \dots \rangle \in B$. By the so-called Wadge's lemma, a winning strategy for player II in $G_L(A, B)$ yields a Lipschitz function witnessing $A \leq_L B$, whereas a winning strategy for player I yields a Lipschitz function witnessing $\omega^\omega \setminus B \leq_L A$. Wadge then assumed determinacy for Lipschitz games as a working hypothesis and he extensively studied the structure of the Lipschitz degrees (i.e. the equivalence classes generated by $\leq_L$) in the Baire space. In

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particular, assuming determinacy, he derived the following somewhat surprising comparability property, known as the *Semi-Linear Ordering principle*:

$$\mathrm{SLO}_L = \text{“For all } A, B \subseteq \omega^\omega\text{, either } A \leq_L B \text{ or } \omega^\omega \setminus B \leq_L A.\text{”}$$

In other words, $\leq_L$ is a linear order provided we identify the degree of a set with that of its complement.

Lipschitz determinacy and SLO_L can be naturally formalized in the language of second order arithmetic (see Section 2). To fix notation, given formula classes Γ_1 and Γ_2, let (Γ_1, Γ_2)-Det_L denote the principle of determinacy for Lipschitz games in the Baire space where player I's pay-off set is Γ_1-definable and player II's pay-off set is Γ_2-definable (if $\Gamma_1 = \Gamma_2$, we will simply write Γ_1-Det_L). Likewise, let (Γ_1, Γ_2)-SLO_L denote the corresponding semi-linear ordering principle.

Two remarkable results on the logical strength of Lipschitz determinacy are to be mentioned. Firstly, A. Louveau and J. Saint Raymond [7] showed that Borel Lipschitz determinacy is provable within second order arithmetic Z_2. Secondly, and very recently, A. Day et al. [2] have shown that the subsystem $\mathsf{ATR}_0 + \Pi^1_1$-induction already proves Borel Lipschitz determinacy. These results evidence that, in the context of second order arithmetic Borel Lipschitz determinacy is a much weaker principle than Borel general determinacy, as Δ^0_4-determinacy for general infinite games is already not provable in Z_2 (see [8]). In the present paper we show that, however, this huge difference in strength does not occur at the initial levels of the Borel hierarchy. Namely,

Theorem 1.

1. *Over* ACA_0*,* Δ^0_1-Det_L*,* Δ^0_1-SLO_L *and* ATR_0 *are pairwise equivalent.*
2. *Over* RCA_0*,* (Δ^0_1, Π^0_1)-Det_L*,* Π^0_1-Det_L*,* $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}_L$ *and* ATR_0 *are pairwise equivalent.*

By a theorem of J. R. Steel [11], ATR_0 is equivalent to clopen and closed determinacy for general infinite games. Hence, for clopen and closed sets Lipschitz and general determinacy are equivalent principles.

Our proof methods have a certain topological flavor. The analysis of the complete sets with respect to the reducibility relation $\leq_L$ developed in [12] can be adapted to prove determinacy of Lipschitz games: roughly speaking, the player who plays in a pay-off set with a richer topological structure will win the game.

The paper is divided into five sections. Section 1 is introductory and Section 2 contains some preliminaries. In Section 3 we study Lipschitz determinacy and SLO_L for clopen sets and obtain a reversal for ATR_0 over the base theory ACA_0. In Section 4 we study Lipschitz determinacy and SLO_L for closed sets and obtain a reversal for ATR_0 over RCA_0. Section 5 contains some concluding remarks.

2 Preliminaries

We assume familiarity with subsystems of second order arithmetic RCA_0, ACA_0 and ATR_0, as presented in [10]. Our notation and terminology are standard and follow [10] (for details and full technical background the reader should consult that book). A formalization of general two-person infinite games within second order arithmetic is described in section V.8 of [10] as well as in section 3 of [9]. (As usual, Γ-Det will denote the principle of general determinacy restricted to Γ games in the Baire space.) A formalization of Lipschitz determinacy and SLO_L in second order arithmetic can be found in [6] or in section 2 of [1]. Here we restrict ourselves to presenting some basic notions and terminology that will be used extensively in this paper.

Within RCA_0, we define $\mathbb{N}$ to be the unique set X such that $\forall i\,(i \in X)$ and we define a numerical pairing function by letting $(i,j) = (i+j)^2 + i$. Using Δ^0_1 comprehension, we can prove that for all sets $X, Y \subseteq \mathbb{N}$, there exists a set $X \times Y \subseteq \mathbb{N}$ consisting of all (i,j) such that $i \in X$ and $j \in Y$. A function $f : X \to Y$ is defined to be a set $f \subseteq X \times Y$ such that for all $i \in X$ there is exactly one $j \in Y$ such that $(i,j) \in f$ (we will also write $f \in Y^X$). For $i \in X$, $f(i)$ is defined to be the unique j such that $(i,j) \in f$. Finite sequences of natural numbers can be encoded as a single natural number and this coding can be developed formally within RCA_0. The set of all (codes of) finite sequences from X is denoted $X^{<\mathbb{N}}$. The *empty sequence* is denoted $\langle\rangle$. Given any $s, t \in X^{<\mathbb{N}}$, $|s|$ denotes the length of s, $s(i)$ or $(s)_i$ denotes the $(i+1)$-th element of s for $i < |s|$, and, for each $j \leq |s|$, $s\,[j]$ is the *j-th initial segment of s*, i.e. $\langle s(0), \dots, s(j-1)\rangle$. If $s = t\,[j]$ for some $j \leq |t|$, we write $s \subseteq t$ and say that s is an initial segment of t (or t is an extension of s). The concatenation of s and t, written $s * t$, is the sequence $\langle s(0), \dots, s(|s|-1), t(0), \dots, t(|t|-1)\rangle$. If $f \in X^{\mathbb{N}}$, $s * f$ denotes $\langle s(0), \dots, s(|s|-1), f(0), f(1), \dots\rangle$, and $f\,[j]$ denotes $\langle f(0), \dots, f(j-1)\rangle$. If $s = f\,[|s|]$, we write $s \subset f$ and say that s is an initial segment of f (or f is an extension of s). If s and t are sequences with $|s| = |t|$, $s \otimes t$ denotes the sequence of length $2|s|$ where $(s \otimes t)_{2i} = (s)_i$ and $(s \otimes t)_{2i+1} = (t)_i$ if $0 \leq i < |s|$.

Consider formulas $A(f)$ and $B(g)$ with distinguished function variables $f, g \in \mathbb{N}^{\mathbb{N}}$. A *Lipschitz game in the Baire space*, denoted $G_L(A,B)$, is defined as follows: Two players, say player I (male) and player II (female), alternately choose an element x in $\mathbb{N}$ to form the resulting plays $f = \langle x_0, x_1, x_2, \dots\rangle \in \mathbb{N}^{\mathbb{N}}$ and $g = \langle y_0, y_1, y_2, \dots\rangle \in \mathbb{N}^{\mathbb{N}}$:

Player I	x_0		x_1		x_2		$\dots$	
Player II		y_0		y_1		y_2		$\dots$

Player II wins just in case $A(f) \leftrightarrow B(g)$ holds. Put $\mathrm{Seq}_{\mathsf{even}} = \{s \in \mathbb{N}^{<\mathbb{N}} : |s| \text{ is even}\}$ and $\mathrm{Seq}_{\mathsf{odd}} = \{s \in \mathbb{N}^{<\mathbb{N}} : |s| \text{ is odd}\}$. A *strategy for player* I in the game $G_L(A,B)$ is a function $\sigma_{\mathrm{I}} : \mathrm{Seq}_{\mathrm{even}} \to \mathbb{N}$ and a *strategy for player* II is a function $\sigma_{\mathrm{II}} : \mathrm{Seq}_{\mathrm{odd}} \to \mathbb{N}$. If players I and II follow strategies σ_{I} and σ_{II}, respectively, the resulting plays are uniquely determined. We will write $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}}$ to denote player I's resulting play and write $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}}$ to denote player II's resulting play. A strategy for a player is a *winning strategy* if the player wins the game as long as he/she plays following it, no matter what his/her opponent plays. A game is *determined* if either player I or player II has a winning strategy. The following axiom, denoted $\mathsf{Det}_L(A,B)$, expresses that the Lipschitz game

$G_L(A,B)$ is determined:

$$\exists\sigma_{\mathrm{I}}\,\forall\sigma_{\mathrm{II}}\,\neg(A(\sigma_{\mathrm{I}}\otimes^{\mathrm{I}}_L\sigma_{\mathrm{II}})\leftrightarrow B(\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_L\sigma_{\mathrm{II}}))\vee\exists\sigma_{\mathrm{II}}\,\forall\sigma_{\mathrm{I}}\,(A(\sigma_{\mathrm{I}}\otimes^{I}_L\sigma_{\mathrm{II}})\leftrightarrow B(\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_L\sigma_{\mathrm{II}})),$$

where σ_{I} and σ_{II} range over strategies for player I and strategies for player II, respectively. Let Γ_1 and Γ_2 be formula classes with distinguished function variables $f,g\in\mathbb{N}^{\mathbb{N}}$, respectively. The scheme of (Γ_1,Γ_2)-*Lipschitz determinacy in the Baire space*, denoted (Γ_1,Γ_2)-Det_L, is given by the axioms $\mathsf{Det}_L(A,B)$, where $A(f)\in\Gamma_1$ and $B(g)\in\Gamma_2$ (if $\Gamma_1=\Gamma_2$, we will simply write Γ_1-Det_L). The scheme of Δ^0_n-*Lipschitz determinacy in the Baire space*, denoted Δ^0_n-Det_L, is given by

$$\forall f\in\mathbb{N}^{\mathbb{N}}(A(f)\leftrightarrow C(f))\wedge\forall g\in\mathbb{N}^{\mathbb{N}}(B(g)\leftrightarrow D(g))\;\rightarrow\;\mathsf{Det}_L(A,B),$$

where $A,B\in\Sigma^0_n$ and $C,D\in\Pi^0_n$. The theories (Γ,Δ^0_n)-Det_L and (Δ^0_n,Γ)-Det_L are defined similarly.

Remark 1. It is easily verified that (Γ_1,Γ_2)-Det_L and $(\neg\Gamma_1,\neg\Gamma_2)$-$\mathsf{Det}_L$ are equivalent over RCA_0, for $G_L(A,B)$ and $G_L(\neg A,\neg B)$ are essentially the same game.

As for SLO_L, the axiom $\mathsf{Red}_L(A,B)\equiv\exists\sigma_{\mathrm{II}}\,\forall\sigma_{\mathrm{I}}\,(A(\sigma_{\mathrm{I}}\otimes^{\mathrm{I}}_L\sigma_{\mathrm{II}})\leftrightarrow B(\sigma_{\mathrm{I}}\otimes^{\mathrm{II}}_L\sigma_{\mathrm{II}}))$ expresses that A *is Lipschitz reducible to* B (i.e., player II has a winning strategy in the game $G_L(A,B)$). The scheme of (Γ_1,Γ_2)-*Lipschitz semilinear ordering principle in the Baire space*, denoted (Γ_1,Γ_2)-SLO_L, is given by the axiom scheme $\mathsf{Red}_L(A,B)\vee\mathsf{Red}_L(\neg B,A)$, where $A\in\Gamma_1$ and $B\in\Gamma_2$. The theories Δ^0_n-SLO_L, (Δ^0_n,Γ)-SLO_L, ... are defined similarly.

Lemma 1. *It is provable over* RCA_0 *that* (Γ_1,Γ_2)-Det_L *implies* (Γ_1,Γ_2)-SLO_L, *and the same holds for classes* (Δ^0_n,Δ^0_m), (Γ,Δ^0_m) *and* (Δ^0_n,Γ).

Proof. See Lemma 5.2 of [1]. □

3 Lipschitz determinacy for clopen sets

First of all we introduce the combinatorial tools we will need to analyse the determinacy of Lipschitz games in the Baire space. A set $T\subseteq X^{<\mathbb{N}}$ is called a *tree* over X if T is closed under initial segments, i.e. $s\in T$ and $t\subseteq s$ imply $t\in T$. We call the elements of T the *nodes* of T. A tree is *infinite* if, for any n, there exists $s\in T$ with $|s|=n$, i.e. if the set of nodes of T is infinite. If $S\subseteq X^{<\mathbb{N}}$ is a tree over X and $S\subseteq T$, then S is called a *subtree* of T.

Fix any tree $T\subseteq X^{<\mathbb{N}}$. A node $s\in T$ is called *terminal* if $\forall a\in X\;(s*\langle a\rangle\notin T)$. A function $f\in X^{\mathbb{N}}$ is called a *path* of T if $\forall n\in\mathbb{N}\,(f\,[n]\in T)$. The set of all paths of T is denoted by $[T]$. A tree $T\subseteq X^{<\mathbb{N}}$ is *well-founded* if it has no path, i.e. $[T]=\emptyset$.

A key fact for the analysis of Lipschitz games is that closed sets in the Baire space correspond to the sets of paths of trees. This fact can be proved in RCA_0 and it is, indeed, an immediate consequence of the normal form theorem for Σ^0_1 formulas (Theorem II.2.7 of [10]).

Proposition 1 ([10], Lemma VI.1.5). *The following is provable in* RCA_0. *Suppose* $X \subseteq \mathbb{N}$. *Assume* $\varphi(f) \in \Pi^0_1$, *with* $f \in X^{\mathbb{N}}$. *Then, there is a tree* $T \subseteq X^{<\mathbb{N}}$ *satisfying that* $[T] = \{f \in X^{\mathbb{N}} : \varphi(f)\}$.

Thus, we identify points in the Baire space with functions $f \in \mathbb{N}^{\mathbb{N}}$, and we identify closed sets in the Baire space with Π^0_1 formulas containing a second order free variable f which ranges over $\mathbb{N}^{\mathbb{N}}$. Similarly, open sets will correspond to Σ^0_1 formulas and so on. We also identify a closed set with the set of paths of a tree, $[T]$, and, by abuse of language, use set theoretic notations to mean the arithmetic formula expressing the corresponding set. (For instance, an expression of the form $f \in [T] - [S]$ is to be understood as the $\Pi^0_1 \wedge \Sigma^0_1$ formula expressing that f is a path of T and is not a path of S.) The following definition is made in RCA_0.

Definition 1. *Given* $X \subseteq \mathbb{N}$, *we say that a tree* $T \subseteq X^{<\mathbb{N}}$ *defines a clopen set if there is a tree* $T' \subseteq X^{<\mathbb{N}}$ *such that* $\forall f \in X^{\mathbb{N}}\,(f \notin [T] \leftrightarrow f \in [T'])$.

The goal of this section is to prove the following reversal for ATR_0 in terms of Lipschitz determinacy and semilinear ordering principle for clopen sets. This result was also obtained in [6] (unpublished).

Theorem 2. *The following are equivalent over* ACA_0:

1. ATR_0.
2. Δ^0_1-Det_L.
3. Δ^0_1-SLO_L.

The remainder of this section is devoted to providing a proof of this result. Our analysis of clopen Lipschitz determinacy rests on basic properties of well-founded trees and ordinal rank functions associated with them. Thus, we shall begin by providing in the next subsection a survey of some basic facts on countable well orderings and well-founded trees that are provable in ATR_0 and that will be needed in the proof of Theorem 2.

3.1 Well-founded trees and ranks

As we mentioned earlier, we identify $\mathbb{N} \times \mathbb{N}$ with a subset of $\mathbb{N}$ using the pairing function $(i, j) = (i + j)^2 + i$. Thus, a binary relation X on $\mathbb{N}$ is identified with a subset of $\mathbb{N} \times \mathbb{N}$. Working over RCA_0 we cannot assume the existence (as sets) of the domain or the range of X. To deal with this difficulty, in RCA_0 an ordering is defined to be a *reflexive* relation (of course, satisfying other additional properties). In RCA_0 we say that the relation $X \subseteq \mathbb{N} \times \mathbb{N}$ is *reflexive* if $\forall i \forall j\,((i, j) \in X \rightarrow ((i, i) \in X \wedge (j, j) \in X))$. If X is reflexive then, by Δ^0_0–comprehension, there exists the set $\mathrm{field}(X) = \{i : \ (i, i) \in X\}$. We also write $i \leq_X j$ for $(i, j) \in X$, and $i <_X j$ for $(i, j) \in X \wedge (j, i) \notin X$.

Within RCA_0, given a reflexive binary relation X, we say that X is *well founded* if there is no $f : \mathbb{N} \rightarrow \mathrm{field}(X)$ such that $f(n+1) <_X f(n)$ for all $n \in \mathbb{N}$.

We say that X is a *countable linear ordering* if it is a reflexive, antisymmetric, transitive, and total relation on its field. We say that X is a *countable well ordering* if it is both well founded and a countable linear ordering.

There is an arithmetical formula $\mathrm{LO}(X)$ expressing that X is a countable linear ordering, and it can be easily checked that there exist Π^1_1 formulas $\mathrm{WF}(X)$ and $\mathrm{WO}(X)$ (with a single free variable X) expressing, respectively, that X is a well founded (reflexive) relation and X is a countable well ordering.

We shall use Greek letters α, β, $\gamma, \dots$ to denote countable well orderings. If α is a well ordering then $\alpha+1$ denotes a well ordering obtained from α by adding an upper bound as follows:

$$\alpha + 1 = \{(2m, 2n) : (m, n) \in \alpha\} \cup \{(1, 1)\} \cup \{(2m, 1) : m \in \mathrm{field}(\alpha)\}.$$

Let us now consider a natural comparability notion between countable well orderings that turns out to be equivalent to Arithmetic Transfinite Recursion.

Definition 2. *Let α and β be countable well orderings. We say that α is weakly less than or equal to β, $\alpha \leq_w \beta$, if there is an injection $f : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ such that $\forall i, j \in \mathrm{field}(\alpha)\ (i \leq_\alpha j \leftrightarrow f(i) \leq_\beta f(j))$.*
We write $\alpha <_w \beta$ if $\alpha + 1 \leq_w \beta$.

Theorem 3 ([5], Theorem 4). *Over RCA_0, ATR_0 is equivalent to the comparability principle: $\forall \alpha, \beta\ (\alpha \leq_w \beta \vee \beta \leq_w \alpha)$.*

The following definitions are made in RCA_0. We follow [5] and section 3 of [3]. Note that for each tree T, the reverse inclusion $\supseteq$ defines a reflexive binary relation on T and T is well-founded if and only if $\supseteq$ is a well-founded relation.

Definition 3. *Let $S, T \subseteq X^{<\mathbb{N}}$ be trees. We shall write $S \preceq T$ if there is a function $f : S \to T$ such that $\forall s_1, s_2 \in S\ (s_1 \subset s_2 \to f(s_1) \subset f(s_2))$.*

Definition 4. *Let $T \subseteq X^{<\mathbb{N}}$ be a tree. A rank function for T is a pair (rk, α) where α is a countable well ordering and $\mathrm{rk} : T \to \mathrm{field}(\alpha)$ satisfies $\alpha = \mathrm{rk}(\langle\rangle) + 1$ and $\mathrm{rk}(t) = \sup\{\mathrm{rk}(s) + 1 : t \subset s \wedge |s| = |t| + 1\}$, for every $t \in T$.*
We say that T is a ranked tree if there exists some rank function for T.

The following basic properties of rank functions can be proved in RCA_0. In particular, from part 2 in the next proposition we see that RCA_0 essentially proves uniqueness of rank functions (see Proposition 3.4 in [3]).

Proposition 2. *The following is provable in RCA_0. Let $T \subseteq X^{<\mathbb{N}}$ be a tree and let (rk, α) be a rank function for T. Then*

1. *$\forall t_1, t_2 \in T\, (t_1 \subset t_2 \to \mathrm{rk}(t_2) <_\alpha \mathrm{rk}(t_1))$.*
2. *If (rk', β) is a rank function for T, then there is an order preserving bijection $h : \mathrm{field}(\alpha) \to \mathrm{field}(\beta)$ such that for all $t \in T$, $\mathrm{rk}(t) = h(\mathrm{rk}'(t))$.*

It can be easily checked that (in RCA_0) every ranked tree is well-founded. The converse can be derived in ATR_0 (this is, essentially, Theorem 7 of [4]):

Theorem 4. *Over* RCA_0*,* ATR_0 *is equivalent to the statement "Every well-founded tree is ranked."*

Rank functions provide a powerful tool in the study of immersions between well-founded trees. The following lemma will be a key ingredient in our proof of Theorem 2.

Lemma 2. *The following is provable in* ACA_0*. Let* $S, T \subseteq \mathbb{N}^{<\mathbb{N}}$ *be ranked trees with rank functions* (rk_1, α) *and* (rk_2, β)*, resp., such that* $S \preceq T$*. Then* $\alpha \leq_w \beta$*.*

Proof. See Lemma 3.7 in [3].

3.2 Proof of Theorem 2

By Theorem V.8.7 in [10] we know that Δ^0_1-Det can be proved in ATR_0 (as a matter of fact, both principles are equivalent over RCA_0). As a consequence, ATR_0 is strong enough to prove determinacy of clopen Lipschitz games, for a Lipschitz game for clopen sets can be effectively reduced to a clopen (general) infinite game. Thus (1) implies (2) and, by Lemma 1, Δ^0_1-Det_L implies Δ^0_1-SLO_L (that is, (2) implies (3)). Therefore, we only must show how to derive ATR_0 from Δ^0_1-SLO_L (working over ACA_0). Let α and β be countable well orderings. We shall prove that $\alpha \leq_w \beta \vee \beta \leq_w \alpha$. By Theorem 3 this suffices to derive ATR_0.

Let $S(\alpha)$ be the tree of decreasing sequences (w.r.t. $<_\alpha$) of elements of field(α)

$$S(\alpha) = \{s \in \mathrm{field}(\alpha)^{<\mathbb{N}} :\ \forall i, j < |s|\ (i < j \to (s)_j <_\alpha (s)_i)\}.$$

Then RCA_0 can prove that $S(\alpha)$ is ranked. Indeed a rank function for $S(\alpha)$ is $\mathrm{rk} : S(\alpha) \to \alpha+1$, defined by (let us note that according to the formal definition of $\alpha + 1$, $1 \in \mathrm{field}(\alpha + 1)$ corresponds to the "ordinal" α)

$$\mathrm{rk}(s) = \begin{cases} 1 & \text{if } s = \langle\rangle \\ (s)_l & \text{if } |s| = l + 1 \end{cases}$$

A similar tree $T(\beta)$ can be defined using β accordingly. Then, as we have remarked, RCA_0 suffices to show that $S(\alpha)$ and $T(\beta)$ are ranked trees and that there are rank functions $(\mathrm{rk}, \alpha+1)$ and $(\mathrm{rk}, \beta+1)$ for $S(\alpha)$ and $T(\beta)$ respectively. Let us define the following trees

$$S = S(\alpha) \cup \{s : \exists t \in S(\alpha)\, \exists t' \in \mathbb{N}^{<\mathbb{N}}\, \exists j\, (t * \langle 2j\rangle \notin S(\alpha) \wedge s = t * \langle 2j\rangle * t')\},$$

$$S' = S(\alpha) \cup \{s : \exists t \in S(\alpha)\, \exists t' \in \mathbb{N}^{<\mathbb{N}}\, \exists j\, (t * \langle 2j+1\rangle \notin S(\alpha) \wedge s = t * \langle 2j+1\rangle * t')\}.$$

Then S and S' are pruned trees (recall that a tree T is said to be pruned if every sequence of T lies on a path of T). In addition, $[S]$ is a clopen set (since $[S']$ corresponds to its complement in the Baire space) and $S \cap S' = S(\alpha)$. In a similar way we define

$$T = T(\beta) \cup \{s : \exists t \in T(\beta)\, \exists t' \in \mathbb{N}^{<\mathbb{N}}\, \exists j\, (t * \langle 2j\rangle \notin T(\beta) \wedge s = t * \langle 2j\rangle * t')\},$$

$T' = T(\beta) \cup \{s : \exists t \in T(\beta)\, \exists t' \in \mathbb{N}^{<\mathbb{N}}\, \exists j\, (t * \langle 2j+1 \rangle \notin T(\beta) \wedge s = t * \langle 2j+1 \rangle * t')\}$.

Once again T and T' are pruned trees, $[T]$ and $[T']$ are clopen sets and $T(\beta) = T \cap T'$.

By Δ^0_1-SLO_L we have $\mathsf{Red}_L([S],[T])$ or $\mathsf{Red}_L([T'],[S])$, since $[T']$ coincides with the complement of $[T]$. If $\mathsf{Red}_L([S],[T])$ holds then player II has a winning strategy σ_{II} in the Lipschitz game $G_L([S],[T])$. In such a case, we define by primitive recursion a function $F : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ putting $F(\langle\rangle) = \langle\rangle$ and

$$F(s * \langle k \rangle) = F(s) * \langle \sigma_{\mathrm{II}}((s \otimes F(s)) * \langle k \rangle) \rangle.$$

Recall that if s and t are sequences with $|s| = |t|$, $s \otimes t$ denotes the sequence of length $2|s|$ where $(s \otimes t)_{2i} = (s)_i$ and $(s \otimes t)_{2i+1} = (t)_i$ if $0 \le i < |s|$. Obviously, if $s_1 \subset s_2$ then $F(s_1) \subset F(s_2)$ and it can be easily checked that

$$\forall s\, (s \in S(\alpha) \to F(s) \in T(\beta)).$$

Indeed, if $s_0 \in S(\alpha)$ but $F(s_0) \notin T(\beta)$ then $F(s_0) \in T - T'$ or $F(s_0) \in T' - T$. Assume $F(s_0) \in T - T'$ (the other case is similar). Then there exists $s' \in S(\alpha)$ such that $s_0 \subseteq s'$ and $s' * \langle 1 \rangle \in S' - S(\alpha)$. Define a strategy σ_{I} for player I as follows:

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} (s')_i \text{ if } |s| = i < |s'| \\ 1 \quad\ \text{otherwise} \end{cases}$$

Then $\sigma_{\mathrm{I}} \otimes^{\mathrm{I}}_L \sigma_{\mathrm{II}} \in [S']$ but $\sigma_{\mathrm{I}} \otimes^{\mathrm{II}}_L \sigma_{\mathrm{II}} \in [T]$. This is a contradiction since σ_{II} is a winning strategy for player II in $G_L([S],[T])$. Thus, using F we show that $S(\alpha) \preceq T(\beta)$ and, by Lemma 2, $\alpha + 1 \le_w \beta + 1$. It easily follows that $\alpha \le_w \beta$.

If $\mathsf{Red}_L([T'],[S])$ holds then there exists a winning strategy for player II in the Lipschitz game $G_L([T'],[S])$, and we can prove reasoning as in the previous case that $T(\beta) \preceq S(\alpha)$ and, as a consequence, $\beta \le_w \alpha$. □

4 Lipschitz determinacy for closed sets

In this section we shall prove new reversals for ATR_0 over the weaker base theory RCA_0. The following definition isolates a notion that will play a key role in our proofs of determinacy within ATR_0.

Definition 5. *We say that a tree $T \subseteq X^{<\mathbb{N}}$ defines a true closed set if*

$$TrueClosed(T) \equiv \exists f \in X^{\mathbb{N}}\, [f \in [T] \wedge \forall k\, \exists s\, (f[k] \subseteq s \wedge s \notin T)].$$

Lemma 3. ACA_0 *proves that if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree then either $TrueClosed(T)$ holds or T defines a clopen set.*

Proof. We work in ACA_0. Suppose that $TrueClosed(T)$ does not hold and define $S = \{s \in \mathbb{N}^{<\mathbb{N}} : \exists t\, (s \subseteq t \wedge t \notin T)\}$. The set S exists by Σ^0_1–comprehension, and it is clear that S is a tree. It is easy to check that $\forall f \in \mathbb{N}^{\mathbb{N}}\, (f \notin [T] \leftrightarrow f \in [S])$ and hence T defines a clopen set. □

Proposition 3. ATR_0 *proves* Π^0_1-Det_L *and* $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}_L$.

Proof. First we show that ATR_0 proves Π^0_1-Det_L. We work in an arbitrary model of ATR_0. Consider $A(f), B(g) \in \Pi^0_1$. By Proposition 1 there are trees $S, T \subseteq \mathbb{N}^{<\mathbb{N}}$ satisfying that $[S] = \{f \in \mathbb{N}^{\mathbb{N}} : A(f)\}$ and $[T] = \{g \in \mathbb{N}^{\mathbb{N}} : B(g)\}$. We must show that the Lipschitz game $G_L([S],[T])$ is determined.

<u>Case 1</u>: $TrueClosed(T)$ holds, i.e., there exists $g_0 \in [T]$ such that $\forall k\, \exists s\, (g_0[k] \subseteq s \wedge s \notin T)$. In this case, there is a winning strategy for player II, σ_{II}, defined as follows: For all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = j$ and $|t| = j$, put

$$\sigma_{\mathrm{II}}((s \otimes t) * \langle n \rangle) = \begin{cases} g_0(j) & \text{if } s * \langle n \rangle \in S \\ \min\{k : t * \langle k \rangle \notin T\} & \text{if } s * \langle n \rangle \notin S \wedge \exists k\, (t * \langle k \rangle \notin T) \\ g_0(j) & \text{if } s * \langle n \rangle \notin S \wedge \forall k\, (t * \langle k \rangle \in T) \end{cases}$$

(In words, player II plays using the boundary point g_0 while player I has played inside S and if player I leaves S at some round then player II will eventually leave T too.) The existence of σ_{II} is granted by ACA_0 and it is straightforward to check that σ_{II} is a winning strategy for player II.

<u>Case 2</u>: Case 1 does not hold but $TrueClosed(S)$ does, i.e., there exists $f_0 \in [S]$ such that $\forall k\, \exists s\, (f_0[k] \subseteq s \wedge s \notin S)$. Put $T' = \{t \in T : \exists t'\, (t \subseteq t' \wedge t' \notin T)\}$. Then, T' exists by arithmetical comprehension, and T' is a well-founded tree since Case 1 fails. Note that if $t_0 \in T - T'$ then we have $\forall t'\, (t_0 \subseteq t' \rightarrow t' \in T)$. Thus, a winning strategy for player I, σ_{I}, can be defined as follows: Let $\sigma_{\mathrm{I}}(\langle\rangle) = f_0(0)$ and for all $s, t \in \mathbb{N}^{<\mathbb{N}}$ with $|s| = |t| = j \geq 1$, put

$$\sigma_{\mathrm{I}}(s \otimes t) = \begin{cases} f_0(j) & \text{if } t \notin T \vee t \in T' \\ \min\{k : s * \langle k \rangle \notin S\} & \text{if } t \in T - T' \wedge \exists k\, (s * \langle k \rangle \notin S) \\ f_0(j) & \text{if } t \in T - T' \wedge \forall k\, (s * \langle k \rangle \in S) \end{cases}$$

Again, σ_{I} exists by ACA_0 and, having in mind that player II must eventually play outside T' since T' is well–founded, it is easy to check that σ_{I} is a winning strategy for player I.

<u>Case 3</u>: Both $TrueClosed(T)$ and $TrueClosed(S)$ fail. By Lemma 3 $[T]$ and $[S]$ are clopen sets and hence $G_L([S],[T])$ is determined by Theorem 2.

We conclude by showing that ATR_0 proves $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}_L$. Given $A(f) \in \Delta^0_1$ and $B_0(g)$, $B_1(g) \in \Pi^0_1$ we show that the Lipschitz game $G_L(A, B_0 \wedge \neg B_1)$ is determined. By Proposition 1 there are trees $S, S', T_0, T_1 \subseteq \mathbb{N}^{<\mathbb{N}}$ satisfying that $[S] = \{f \in \mathbb{N}^{\mathbb{N}} : A(f)\}$, $[S'] = \{f \in \mathbb{N}^{\mathbb{N}} : \neg A(f)\}$ and $[T_i] = \{g \in \mathbb{N}^{\mathbb{N}} : B_i(g)\}$, for $i = 0, 1$. It is easily seen that $S \cap S'$ is a well-founded tree and, without loss of generality, we can assume that $T_1 \subseteq T_0$. We must show that the game $G_L([S], [T_0] - [T_1])$ is determined. Again we distinguish several cases:

<u>Case A</u>: $TrueClosed(T_0)$ does not hold. Then by Lemma 3, T_0 defines a clopen set and there exists a tree T_0' such that $\forall g \in \mathbb{N}^{\mathbb{N}}(g \notin [T_0] \leftrightarrow g \in [T_0'])$. Then

the game $G_L([S],[T_0]-[T_1])$ is equivalent to a game $G_L(A,C)$ for a formula $C(g) \in \Sigma^0_1$, since

$$g \in [T_0]-[T_1] \leftrightarrow g \notin [T'_0] \wedge g \notin [T_1] \leftrightarrow \exists k\,(g[k] \notin T'_0 \wedge g[k] \notin T_1)$$

But let us note that, by Remark 1, Σ^0_1-Det_L is equivalent to Π^0_1-Det_L and so it follows that $G_L([S],[T_0]-[T_1])$ is determined.

Case B: $\mathit{TrueClosed}(T_0)$ holds and there exists some function $g_0 \in [T_0]-[T_1]$ such that $\forall k\,\exists s\,(g_0[k] \subseteq s \wedge s \notin T_0)$. In this case, there is a winning strategy for player II, σ_{II}, defined essentially as in Case 1. Bearing in mind that $S \cap S'$ is well-founded, player II plays using the boundary point g_0 as follows: while player I has played inside $S \cap S'$ player II plays using g_0 and if player I leaves $S \cap S'$ at some round then player II will eventually leave T_0 if player I plays in $S'-(S \cap S')$ or will remain inside $T_0 - T_1$ if player I plays in $S-(S \cap S')$.

Case C: $\mathit{TrueClosed}(T_0)$ holds, but Case B fails. That is, every $g_0 \in [T_0]$ such that $\forall k\,\exists s\,(g_0[k] \subseteq s \wedge s \notin T_0)$ satisfies $g_0 \in [T_1]$. Then, by Arithmetical Comprehension there exists $C = \{t : \forall s\,(t \subseteq s \to s \in T_0)\}$, and, for every $g \in \mathbb{N}^{\mathbb{N}}$,

$$g \in [T_0]-[T_1] \leftrightarrow \exists k\,(g[k] \in C \wedge g[k] \notin T_1)$$

As a consequence, the game $G_L([S],[T_0]-[T_1])$ is again equivalent to a game $G_L(A,D)$, for a formula $D(g) \in \Sigma^0_1$ and, as noted in Case A, is determined. □

Theorem 5. *The following principles are pairwise equivalent over* RCA_0*:*

1. ATR_0.
2. Π^0_1-Det_L.
3. (Δ^0_1, Π^0_1)-Det_L.
4. $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}_L$

Proof. Let us observe that, by Proposition 3, (1) implies (2) and (4). On the other hand, obviously, (2) implies (3) and so we only have to show that both (3) and (4) imply (1).

(3)⇒(1): By Theorem 2 it is sufficient to show that $\mathsf{RCA}_0 + (\Delta^0_1, \Pi^0_1)$-$\mathsf{Det}_L$ implies ACA_0. Assume $\mathsf{RCA}_0 + (\Delta^0_1, \Pi^0_1)$-$\mathsf{Det}_L$ and consider $\varphi(x) \in \Sigma^0_1$ (we disregard parameters). We must show that the set $\{x : \varphi(x)\}$ exists. Write $\varphi(x) \equiv \exists y\,\varphi_0(x,y)$ with $\varphi_0 \in \Delta^0_0$. Define $A(f)$ to be $\forall i \le f(0)\,(f(i) = f(0) - i)$ and $B(g)$ to be

$$\begin{aligned}&\forall l[l = g(0) \to \\ &\quad \forall i \le l\,(g(i) = l - i) \wedge \forall i \le l\,(\exists y\,\varphi_0(i,y) \to \exists y \le g(l{+}1)\,\varphi_0(i,y))]\end{aligned}$$

That is to say, a play for player I is in A if it is of the form

$$\langle k, (k-1), (k-2), \ldots, 0\rangle * f'$$

for some $k \in \mathbb{N}$ and $f' \in \mathbb{N}^{\mathbb{N}}$. A play for player II is in B if it is of the form

$$\langle l, (l-1),\ (l-2),\ \ldots,\ 0\rangle * \langle m\rangle * g'$$

for some $l, m \in \mathbb{N}$ and $g' \in \mathbb{N}^{\mathbb{N}}$ and, in addition, for each $i \le l$, if $\varphi(i)$ holds then $\exists y \le m\,\varphi_0(i,y)$ holds too. Note that $A(f) \in \Delta^0_1$ and $B(g) \in \Pi^0_1$.

Claim. Player I cannot have a winning strategy in the game $G_L(A, B)$.

Proof. Towards a contradiction, assume that σ is a winning strategy for player I and fix $k_0 = \sigma(\langle\rangle)$. By the strong Σ^0_1 bounding scheme (which is available in RCA_0), there exists m_0 satisfying that $\forall i \leq k_0\, (\exists y\, \varphi_0(i, y) \to \exists y \leq m_0\, \varphi_0(i, y))$. Now consider a strategy for player II, τ, defined as follows: Player II mimics player I's first k_0+1 moves and in her (k_0+2)-th move, player II picks m_0. Clearly, we have $(A(\sigma \otimes^{\mathrm{I}} \tau) \leftrightarrow B(\sigma \otimes^{\mathrm{II}} \tau))$, which contradicts the fact that σ is a winning strategy for player I. □

In view of the previous claim, it follows from (Δ^0_1, Π^0_1)-Det_L that player II has a winning strategy, say σ_0, in $G_L(A, B)$. For each $k \in \mathbb{N}$, let σ^k_{I} denote the strategy for player I according to which player I plays as follows:

$$\langle k+1,\, k,\, (k-1),\, (k-2),\, \ldots,\, 0\rangle * \langle 0,\, 0,\, 0, \ldots\rangle$$

It is clear that $A(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{I}} \sigma_0)$ holds and hence $B(\sigma^k_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_0)$ holds as well, for σ_0 is a winning strategy for player II. Put $g = \sigma^k_{\mathrm{I}} \otimes^{\mathrm{II}} \sigma_0$, $l = g(0)$ and $m = g(l+1)$. Then, we have $\forall i \leq l\, (\exists y\, \varphi_0(i, y) \to \exists y \leq m\, \varphi_0(i, y))$. But observe that $k \leq l$ (for otherwise it is easy to construct a strategy for player I that would allow player I to beat player II's strategy σ_0). As a result, we have $\varphi(k) \leftrightarrow \exists y \leq m\, \varphi_0(k, y)$. By Δ^0_1-comprehension, there exists $S \subseteq \mathrm{Seq}_{\mathrm{even}} \times \mathbb{N} \times \mathbb{N}$ such that $(S)_k = \sigma^k_{\mathrm{I}}$ for each k, where $(S)_k = \{(s, n) \in \mathrm{Seq}_{\mathrm{even}} \times \mathbb{N} : (s, n, k) \in S\}$. Then, for each $k \in \mathbb{N}$ we have

$$\varphi(k) \leftrightarrow \exists l, m\, (l = ((S)_k \otimes^{\mathrm{II}} \sigma_0)(0) \wedge m = ((S)_k \otimes^{\mathrm{II}} \sigma_0)(l{+}1) \wedge \exists y \leq m\, \varphi_0(k, y))$$

and

$$\varphi(k) \leftrightarrow \forall l, m\, (l = ((S)_k \otimes^{\mathrm{II}} \sigma_0)(0) \wedge m = ((S)_k \otimes^{\mathrm{II}} \sigma_0)(l{+}1) \to \exists y \leq m\, \varphi_0(k, y))$$

Thus, the set $\{x : \varphi(x)\}$ exists by Δ^0_1-comprehension.

(4)⇒(1): By Theorem 2 it suffices to show that $\mathsf{RCA}_0 + (\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}_L$ implies ACA_0. To this end, we will adapt the proof of (3)⇒(1). Assume $\mathsf{RCA}_0 + (\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}_L$ and consider $\varphi(x) \in \Sigma^0_1$. Write $\varphi(x) \equiv \exists y\, \varphi_0(x, y)$ with $\varphi_0 \in \Delta^0_0$. Define $A(f)$ to be $\forall i \leq f(0)\, (f(i) = f(0) - i)$ and $B'(g)$ to be

$$\exists l\, [g(l) = 1 \wedge \forall l' < l\, (g(l') = 0)] \wedge$$
$$\forall l\, [g(l) = 1 \wedge \forall l' < l\, (g(l') = 0) \to \forall i \leq l\, (\exists y\, \varphi_0(i, y) \to \exists y \leq g(l{+}1)\, \varphi_0(i, y))]$$

That is to say, a play for player I is in A if it is of the form

$$\langle k, (k-1), (k-2), \ldots, 0\rangle * f'$$

for some $k \in \mathbb{N}$ and $f' \in \mathbb{N}^{\mathbb{N}}$. A play for player II is in B' if it is of the form

$$\langle\overbrace{0, \ldots, 0}^{l-1 \text{ times}}\rangle * \langle 1\rangle * \langle m\rangle * g'$$

for some $l, m \in \mathbb{N}$ and $g' \in \mathbb{N}^{\mathbb{N}}$ and, in addition, for each $i \leq l$, if $\varphi(i)$ holds then $\exists y \leq m\, \varphi_0(i, y)$ holds too. Note that $A(f) \in \Delta^0_1$ and $B'(g) \in \Sigma^0_1 \wedge \Pi^0_1$.

Reasoning as in the proof of (3)⇒(1), one can show that player II cannot have a winning strategy in the game $G_L(\neg B', A)$. Hence, by $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{SLO}_L$ player II has a winning strategy in the game $G_L(A, B')$, say σ_0. Again reasoning as in the proof of (3)⇒(1), one can show that $\{x : \varphi(x)\}$ exists by Δ^0_1-comprehension using σ_0 as a parameter. □

5 Concluding remarks

This paper studies the logical strength of Lipschitz determinacy for the first levels of the Borel hierarchy in the Baire space in terms of subsystems of second order arithmetic. Two natural questions for future research arise in this context. We know from [2] that full Borel Lipschitz determinacy is provable within the subsystem $\mathsf{ATR}_0 + \Pi^1_1$-induction, while we have shown here that ATR_0 suffices for proving $(\Delta^0_1, \Sigma^0_1 \wedge \Pi^0_1)$-$\mathsf{Det}_L$. But, (*Q1*) what is the highest level (Γ_1, Γ_2) for which (Γ_1, Γ_2)-Det_L remains provable in ATR_0? (*Q2*) What is the smallest level (Γ_2, Γ_3), if any, for which (Γ_2, Γ_3)-Det_L implies $\mathsf{ATR}_0 + \Pi^1_1$-induction over an appropriate base theory?

In [12], Wadge also introduced the so-called *Wadge games* (a variation of Lipschitz games where player II is allowed to pass) to analyze reducibility via continuous functions in the Baire space. A natural line for future work would involve calibrating the logical strength of Wadge determinacy and Wadge SLO for different levels of the Borel hierarchy in the Baire space. Some progress in this direction has already been made in [6], and [1] provides an analysis of both Lipschitz and Wadge determinacy for the initial levels of the Borel hierarchy in the Cantor space.

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